THE DISTRIBUTION OF THE MINIMUM DISTANCE BETWEEN A RANDOM TARGET AND UNITS PATROLLING ALONG A LINE

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NAVAL POSTGRADUATE SCHOOL
Monterey, California

THESIS

THE DISTRIBUTION OF THE
MINIMUM DISTANCE BETWEEN A RANDOM TARGET
AND UNITS PATROLLING ALONG A LINE

by

Joseph Dallas Clarke IV

March 1979

Thesis Advisor: D. Barr

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(20. ABSTRACT Continued)

lattice of fixed points will form upper and lower bounds for the more algebraically tedious two-dimensional area patrol model. The hypothetical bounding distributions are developed for n units patrolling an area. Finally a one dimensional radius of influence model is developed which quantifies the contribution that the effective operational radius of a carrier airwing makes to the initial minimum distance analytic model.

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The Distribution of the Minimum Distance Between a Random Target and Units Patrolling Along a Line

by

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ABSTRACT

The probability distribution of the minimum of the distances from a randomly occurring trouble spot to n carriers patrolling along a line of length L is analyzed. Two approximations, a Poisson process and a fixed lattice of equally spaced points, which bound the analytic model are developed and their usefulness and limitations are discussed. It is hypothesized that a two dimensional Poisson field and a two dimensional lattice of fixed points will form upper and lower bounds for the more algebraically tedious two-dimensional area patrol model. The hypothetical bounding distributions are developed for n units patrolling an area. Finally a one dimensional radius of influence model is developed which quantifies the contribution that the effective operational radius of a carrier airwing makes to the initial minimum distance analytic model.

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I. INTRODUCTION

The Commander in Chief Pacific has the basic responsibility of exercising control of the military forces of the United States from the west coast of North America to the east coasts of Asia and Africa. Since most of this vast area is either ocean or accessible by sea lines of communication, the U.S. Navy has a critical role in the maintenance of a U.S. military presence in the Pacific Command area.

The problem addressed in this thesis was posed by Vice Admiral E.P. Aurand, USN to Karl Eulenstein, Senior Analyst in Section J 77 at CINCPAC. As COMTHIRDFLT, Admiral Aurand wanted to know how many aircraft carriers would be required to patrol the western littoral of the Pacific Ocean Basen as well as the littorals of the Indian Ocean Basen in order that at least one patrolling carrier would be within a command specified distance of a randomly occurring trouble spot with at least a command specified probability. More formally, what is the probability distribution of the minimum of the distances between n carriers and a single random trouble spot somewhere along the patrolled coast

This thesis addresses the analytical expression of,

$$P_n[D = \min(D_1, D_2, \ldots, D_n) \leq d] \qquad (1)$$

where:

n is the number of carriers on station

$D_1, D_2 \ldots D_n$ are the distances from each carrier to a random trouble spot

d is a command specified distance measured from the trouble spot.

The model allows the determination of the minimum n required in expression (1) such that the probability is at least as large as p. Other related inferences and applications are discussed.

In the following a one-dimensional model and two approximations that give useful bounds to it are developed. Then the analogous two-dimensional bounding models are considered. Finally the concept of a "sphere of influence" is incorporated into the one-dimensional model.

II. INITIAL ANALYTIC SOLUTION

A. ASSUMPTIONS

To increase the tractability of the analysis several simplifying assumptions were made:

(1) The complex curvilinear coastlines which mark the western boundry of CINCPAC'S area of responsibility are approximated by a straightline of length L.

(2) Each of the n carriers were assigned individual patrol line segments of length $\frac{L}{n}$ miles.

(3) A carrier's position is assumed to be uniformly distributed on its assigned patrol line segment.

(4) A trouble spot will appear at random along the coast line.

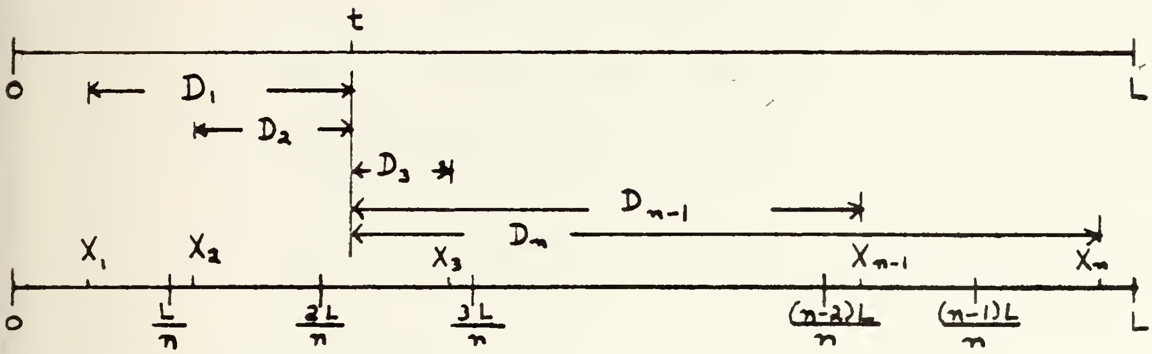
(5) An individual carrier is considered as a point. (The effects the carrier airwing's radius of influence are analysed in Chapter V.)

(6) Each carrier's location within its patrol segment is independent of the locations of the other n-1 carriers. The location of a trouble spot T is independent of the locations of the carriers. See Figure 1.

B. DEVELOPMENT

The Mean Value Theorem for Integrals is utilized in the derivation of the analytical results detailed below. This theorem shows one may approximate a probability density function at a point h as

Schematic and Symbology for the Analytic Model



X_i, $i = 1,2, \ldots, n$	the location of carrier i within line segment i
D_i, $i = 1,2, \ldots, n$	the distance to carrier i measured from a randomly occurring trouble spot t
L	length of line to be patrolled
n[1]	the number of carriers assigned to patrol a line of length L
t	randomly occurring trouble spot

[1]Notationally this paper will follow the custom of using uppercase letters for random variables and lowercase letters for parameters and outcomes with the exception of the parameter L which will denote the length of the line to be patrolled.

Figure 1

$$f(h)\Delta h = \lim_{\Delta h \to 0^+} P[X \varepsilon (h, h+\Delta h)] .$$

Examination of the geometry of the problem suggests that the development of the cumulative distribution function, $F_D(d)$[1] $= P_n[D = \min(D_1, \ldots, D_n) \leq d]$ be segmented as follows:

(1) Left end segment such that $T \varepsilon (0, \frac{L}{n})$

(2) Middle segments such that $T \varepsilon (\frac{(i-1)L}{n}, \frac{iL}{n})$ for some $i = 2, 3, \ldots, n-1$

(3) Right end segment such that $T \varepsilon (\frac{(n-1)L}{n}, L)$

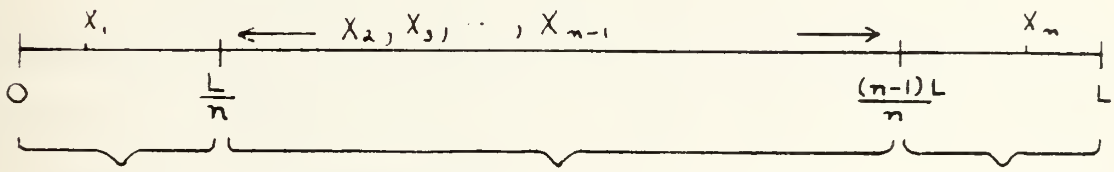


Figure 2. Line segmentation

The general analytic result was developed by conditioning on the segment containing the target. As a result of the assumption that the trouble spot will appear at random along the line (0,L), the target is located in segment i with

[1]The cumulative distribution function which will be developed is actually a function of the minimum of the distances, d; the length of the patrolled line, L; and the number of patrolling units, n; $F_D(d;L,n)$. For simplicity this terminology will be abbreviated as $F_D(d)$.

probability 1/n; that is,

$$P[T \varepsilon [\frac{(i-1)n}{L},\frac{in}{L}]] = \frac{1}{n}; \quad \text{for} \quad i = 1,2, \ldots, n. \tag{2}$$

Given that $T \varepsilon [\frac{(i-1)n}{L},\frac{in}{L}]$, we further condition on T being located in the first or second half of this line segment. Let $D_{i-1} = |T - X_{i-1}|$, $D_i = |T - X_i|$, $D_{i+1} = |T - X_{i+1}|$, and $D = [\min(D_{i-1},D_i, D_{i+1})]$; where D is the random variable whose cdf is being developed. The conditional pdf of D given T is unconditional over T (being located in a specific half of a line segment) and integrated over the range (0,d) to determine the conditioned cumulative distribution function (cdf) of $D = [\min(D_{i-1},D_i,D_{i+1})]$ given that T is an element of a specific line segment i.

1. Left-hand Segment

The assumptions made in developing this portion of the model are:

(1) $T \sim \text{Uniform } (0,L)$, so $T \varepsilon (0,\frac{L}{n})$ with probability 1/n

(2) $X_1 \sim \text{Uniform } (0,\frac{L}{n})$

(3) $X_2 \sim \text{Uniform } (\frac{L}{n}, \frac{2L}{n})$

(4) X_1, X_2, T are independent random variables.

a. Conditional Probability Density Function

Let S(i) be the event that the random variable T occurs in the i^{th} segment $(\frac{(i-1)L}{n},\frac{iL}{n})$ of the line; then $P[S(i)] = 1/n$.

Consider the case $t \varepsilon (0, \frac{L}{2n})$, so

$$D = D_1 = |T - X_1|,$$

and the maximum value of D is $\frac{L}{n} - t$. For $0 \leq d \leq t$,

$$f_{D|T,S(i)}(d|t)\Delta d \approx P[D \varepsilon (d, d+\Delta d)]$$

$$= P[X_1 \varepsilon (t-d, t-d-\Delta d) \text{ or } X_1 \varepsilon (t+d, t+d+\Delta d)]$$

$$= 2\Delta dn/L .$$

For $t \leq d \leq \frac{L}{n} - t$,

$$P[D \varepsilon (d, d+\Delta d)] = P[X_1 \varepsilon (t+d, t+d+\Delta d)] = \Delta dn/L.$$

Consider next the case $t \varepsilon (\frac{L}{2n}, \frac{L}{n})$, so $D = \min(D_1, D_2)$ and $D \leq t$. For $0 \leq d \leq \frac{L}{n} - t$,

$$P[D \varepsilon (d, d+\Delta d)] = P[X_1 \varepsilon (t-d-\Delta d, t-d)$$

$$\text{or } X_1 \varepsilon (t+d, t+d+\Delta d)] = 2\Delta dn/L .$$

For $\frac{L}{n} - t \le d \le t$,

$$P[D \,\varepsilon\, (d,d+\Delta d)] = P[\{X_1 \,\varepsilon\, (t-d-\Delta d, t-d) \text{ and } X_2 > t+d+\Delta d\}$$

$$\text{or } \{X_1 < t-d-\Delta d \text{ and } X_2 \,\varepsilon\, (t+d, t+d+\Delta d)\}]$$

$$= (\Delta dn/L)\cdot(\frac{2L}{n} - t - d)\cdot\frac{n}{L} + ((t-d)\cdot\frac{n}{L})\cdot(\frac{\Delta dn}{L})$$

$$= \Delta d(\frac{n}{L})^2[\frac{2L}{n} - t - d - \Delta d + (t-d-\Delta d)]$$

$$= 2\Delta dn/L \cdot [1 - \frac{n(d+\Delta d)}{L}] \; .$$

Thus it is seen that:

$$f_{D|T}(d|t) = \begin{cases} 2n/L \; ; & 0 \le d \le t \\ n/L \; ; & t \le d \le (\frac{L}{n}) - t \end{cases} \quad \text{for } t \,\varepsilon\, (0,\frac{L}{2n})$$

$$= \begin{cases} \frac{2n}{L} \; ; & 0 \le d \le (\frac{L}{n}) - t \\ \frac{2n}{L}(1 - \frac{nd}{L}) \; ; & (\frac{L}{n}) - t \le d \le t \end{cases} \quad \text{for } t \,\varepsilon\, (\frac{L}{2n},\frac{L}{n}) \; . \tag{3}$$

b. Unconditioning Over $t \varepsilon (0, \frac{L}{2n})$ And $t \varepsilon (\frac{L}{2n}, \frac{L}{n})$ Given S(1)

The conditional probability density function is graphed below, where values of $f_{D|T}(d|t)$ are shown for various regions of the d-t plane.

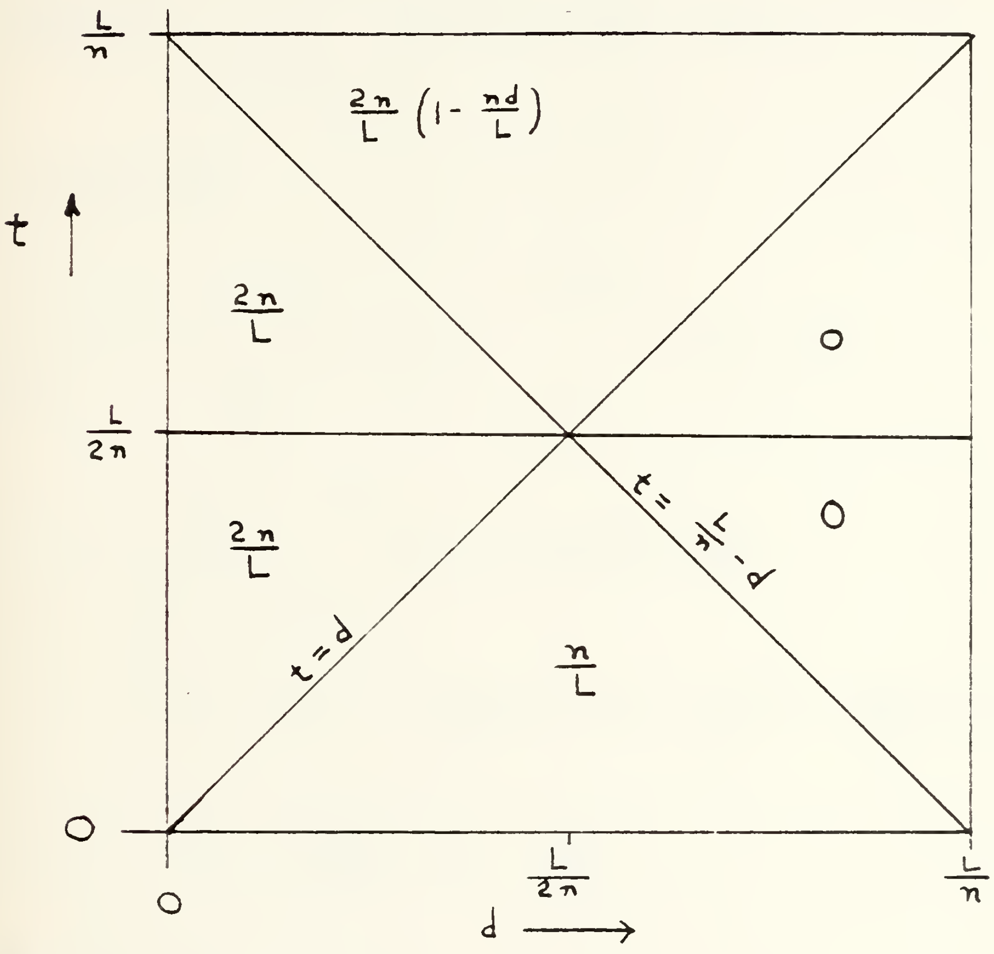


Figure 3. Unconditional probability density function

We now find the conditional distribution of D, given S(1), by taking the expectation of $f_{D|T,S(1)}(d|t)$ with respect to the conditional distribution of T given S(1). Thus unconditioning over T (given S(1)) has the mathematical form:

$$f_{D|S(1)}(d) = \int_0^{L/n} f_{D|T,S(1)}(d|t)\, f_{T|S(1)}(t)\, dt \ .$$

For $0 \le d \le L/2n$,

$$f_{D|S(1)}(d) = \int_0^{d} \frac{n}{L} \cdot \frac{n}{L}\, dt + \int_d^{\frac{L}{n}-d} \frac{2n}{L} \cdot \frac{n}{L}\, dt$$

$$+ \int_{\frac{L}{n}-d}^{\frac{L}{n}} \frac{2n}{L} \cdot (1 - \frac{nd}{L}) \cdot \frac{n}{L}\, dt$$

$$= (\frac{n}{L})^2 d + 2(\frac{n}{L})^2 (\frac{L}{n} - 2d) + 2(\frac{n}{L})^2 (1 - \frac{nd}{L}) (\frac{L}{n} - \frac{L}{n} + d)$$

$$= (\frac{n}{L})^2 [\frac{2L}{n} - d - 2nd^2/L] \ .$$

For $\frac{L}{2n} \le d \le \frac{L}{n}$,

$$f_{D|S(1)}(d) = \int_0^{\frac{L}{n}-d} \frac{n}{L} \cdot \frac{n}{L}\, dt + \int_d^{\frac{L}{n}} \frac{2n}{L}(1 - \frac{nd}{L}) \cdot \frac{n}{L}\, dt$$

$$= (\frac{n}{L})^2 (\frac{L}{n} - d) + 2(\frac{n}{L})^2 (1 - \frac{nd}{L}) (\frac{L}{n} - d)$$

$$= (\frac{n}{L})^2 (\frac{3L}{n} - 5d + 2d^2 n/L) .$$

Therefore in the lefthand segment,

$$f_{D|S(1)}(d) = \begin{cases} \left(\frac{n}{L}\right)^2\left[\frac{2L}{n} - d - 2nd^2/L\right]; & \text{for} \quad 0 \le d \le \frac{L}{2n} \\ \left(\frac{n}{L}\right)^2\left[\frac{3L}{n} - 5d + 2d^2n/L\right]; & \text{for} \quad \frac{L}{2n} \le d \le \frac{L}{n} . \end{cases} \tag{4}$$

c. The Conditional Cumulative Distribution Function

The cumulative distribution function (cdf) of D conditioned by S(l) in accordance with expression (2) is calculated by integrating the pdf over the range (0,d) of D:

$$F_{D|S(1)}(d) = \int_0^d f_{D|S(1)}(h)\, dh .$$

For $0 \le d \le \frac{L}{2n}$,

$$F_{D|S(1)}(d) = \int_0^d \left(\frac{n}{L}\right)^2\left[\frac{2L}{n} - h - 2nh^2/L\right] dh$$

$$= \left(\frac{n}{L}\right)^2\left[\frac{2Ld}{n} - \frac{d^2}{2} - \frac{2nd^3}{3L}\right] , \text{ and}$$

$$F_{D|T}\left(\frac{L}{2n}\right) = \frac{19}{24} .$$

For $\frac{L}{2n} \leq d \leq \frac{L}{n}$,

$$F_{D|T}(d|t) = \int_{L/2n}^{d} \left(\frac{n}{L}\right)^2 \left[\frac{3L}{n} - 5h + 2h^2 \frac{n}{L}\right] dh + \frac{19}{24}$$

$$= \left(\frac{n}{L}\right)^2 \left[\frac{3Ld}{n} - \frac{5d^2}{2} + \frac{2nd^3}{3L} - \left(\frac{3L}{n}\right)\left(\frac{L}{2n}\right) + \left(\frac{5}{2}\right)\left(\frac{L}{2n}\right)^2 - \left(\frac{2n}{3L}\right)\left(\frac{L}{2n}\right)^3 + \frac{19}{24}\right.$$

$$= \left(\frac{n}{L}\right)^2 \left[\frac{3Ld}{n} - \frac{5d^2}{2} + \frac{2nd^3}{3L} - \frac{23L^2}{24n^2} + \frac{19}{24}\right] .$$

For the left-hand segment,

$$F_{D|S(1)}(d) = \begin{cases} \left(\frac{n}{L}\right)^2 \left[\frac{2Ld}{n} - \frac{d^2}{2} - \frac{2nd^3}{3L}\right]; & \text{for } 0 \leq d \leq \frac{L}{2n} \\ \frac{19}{24} + \left(\frac{n}{L}\right)^2 \left[\frac{3Ld}{n} - \frac{5d^2}{2} + \frac{2nd^3}{3L} - \frac{23L^2}{24n^2}\right]; & \text{for } \frac{L}{2n} \leq d \leq \frac{L}{n} \\ 1 ; & \text{for } d \geq \frac{L}{n} . \end{cases} \qquad (5)$$

2. <u>Middle Line Segments</u>

We next develop the cumulative distribution functions for the minimum of the distances between a random trouble spot and n-2 carriers assigned to patrol stations i = 2,3, ..., n-1; stations which have at least one other patrol unit on either side of their patrol segment.

The middle line segment conditional cdf is identical for each of the "middle" patrol stations and is distinguished from that associated with either of the end patrol stations as follows:

(1) end segments

$$D = \min(D_1, D_2) \quad \text{or} \quad D = \min(D_{n-1}, D_n) .$$

(2) middle line segments

$$D = \min(D_{i-1}, D_i, D_{i+1}) \quad \text{for} \quad i = 2,3, \ldots, n-1 .$$

A typical middle line segment is segment 2, $(\frac{L}{n}, \frac{2L}{n})$. The assumptions for developing this portion of the model are:

(1) $T \sim$ Uniform $(0,L)$, so $T \in (\frac{L}{n}, \frac{2L}{n})$ with probability $\frac{1}{n}$,

(2) $X_1 \sim$ Uniform $(0, \frac{L}{n})$,

(3) $X_2 \sim$ Uniform $(\frac{L}{n}, \frac{2L}{n})$,

(4) $X_3 \sim$ Uniform $(\frac{2L}{n}, \frac{3L}{n})$

(5) X_1, X_2, X_3, and T are independent random variables.

a. Conditional Probability Density Function

Consider a given $t \in (\frac{L}{n}, \frac{3L}{2n})$, so

$$D_1 = |t - X_1| ,$$

$$D_2 = |t - X_2| ,$$

$$D = \min\{D_1, D_2\} ,$$

and the maximum value of D is $\frac{2L}{n} - t$. For $0 \leq d \leq t - \frac{L}{n}$,

$$f_{D|T,S(2)}(d|t)\Delta d \;\simeq\; P[D \in (d,d+\Delta d)]$$

$$= \; P[X_2 \in (t-d,t-d-\Delta d) \text{ or } X_2 \in (t+d,t+d+\Delta d)] \;=\; 2\Delta dn/L.$$

For $t - \frac{L}{n} \leq d \leq \frac{2L}{n} - t$,

$$P[D \in (d,d+\Delta d)] \;=\; P[\{X_1 \in (t-d,t-d-\Delta d) \text{ and } X_2 > (t+d+\Delta d)\}$$

$$\text{or } \{X_2 \in (t+d,t+d+\Delta d) \text{ and } X_1 < (t-d-\Delta d)\}]$$

$$= \frac{dn}{L} \cdot [\frac{2L}{n} - (t+d+\Delta d)] \cdot \frac{n}{L} + \frac{\Delta dn}{L}[t-d-\Delta d]\cdot\frac{n}{L}$$

$$= \frac{2\Delta dn^2}{L^2}[\frac{L}{n} - d - \Delta d]$$

$$= \frac{2\Delta dn}{L}[1 - (d+\Delta d)\frac{n}{L}] \;.$$

Consider next a given $t \in (\frac{3L}{2n},\frac{2L}{n})$, so $D_2 = |t - X_2|$, $D_3 = |t - X_3|$, $D = \min(D_2,D_3)$ and the maximum value of D is $t - \frac{L}{n}$.

For $0 \leq d \leq \frac{2L}{n} - t$,

$$P[D \varepsilon (d,d+\Delta d)] = P[X_2 \varepsilon (t+d,t+d+\Delta d) \text{ or } X_2 \varepsilon (t-d,t-d-\Delta d)]$$

$$= \frac{2\Delta dn}{L}.$$

For $\frac{2L}{n} - t \leq d \leq t - \frac{L}{n}$,

$$P[D \varepsilon (d,d+\Delta d)] = P[\{X_2 \varepsilon (t-d,t-d-\Delta d) \text{ and } X_3 > (t+d \ \Delta d)\} \text{ or }$$

$$\{X_3 \varepsilon (t+d,t+d+\Delta d) \text{ and } X_2 < (t-d-\Delta d)\}]$$

$$= \frac{\Delta dn}{L} \cdot [\frac{3L}{n} - (t+d+\Delta d)]\frac{n}{L} + \frac{\Delta dn}{L}(t-d-\Delta d-\frac{L}{n})\frac{n}{L}$$

$$= \frac{2\Delta dn^2}{L^2}[\frac{L}{n} - d - \Delta d]$$

$$= \frac{2\Delta dn}{L}[1 - (d+\Delta d)n/L].$$

Thus it is seen that:

$$f_{D|T,S(2)}(d|t) = \begin{cases} \frac{2n}{L}; & 0 \leq d \leq t - \frac{L}{n} \\ \frac{2n}{L}[1 - \frac{dn}{L}]; & t - \frac{L}{n} \leq d \leq \frac{2L}{n} - t \end{cases} \quad t \varepsilon (\frac{L}{n}, \frac{3L}{2n})$$

$$\qquad \begin{cases} \frac{2n}{L}; & 0 \leq d \leq \frac{2L}{n} - t \\ \frac{2n}{L}[1 - \frac{nd}{L}]; & \frac{2L}{n} - t \leq d < t - \frac{L}{n} \end{cases} \quad t \varepsilon (\frac{3L}{2n}, \frac{2L}{n}).$$

(6)

b. Unconditioning Over $t \varepsilon (\frac{L}{n}, \frac{3L}{2n})$ And $t \varepsilon (\frac{3L}{2n}, \frac{2L}{n})$ Given S(2)

The conditional probability density function is graphed below, where values of $f_{D|T}(d|t)$ are given for various regions of the d-t plane.

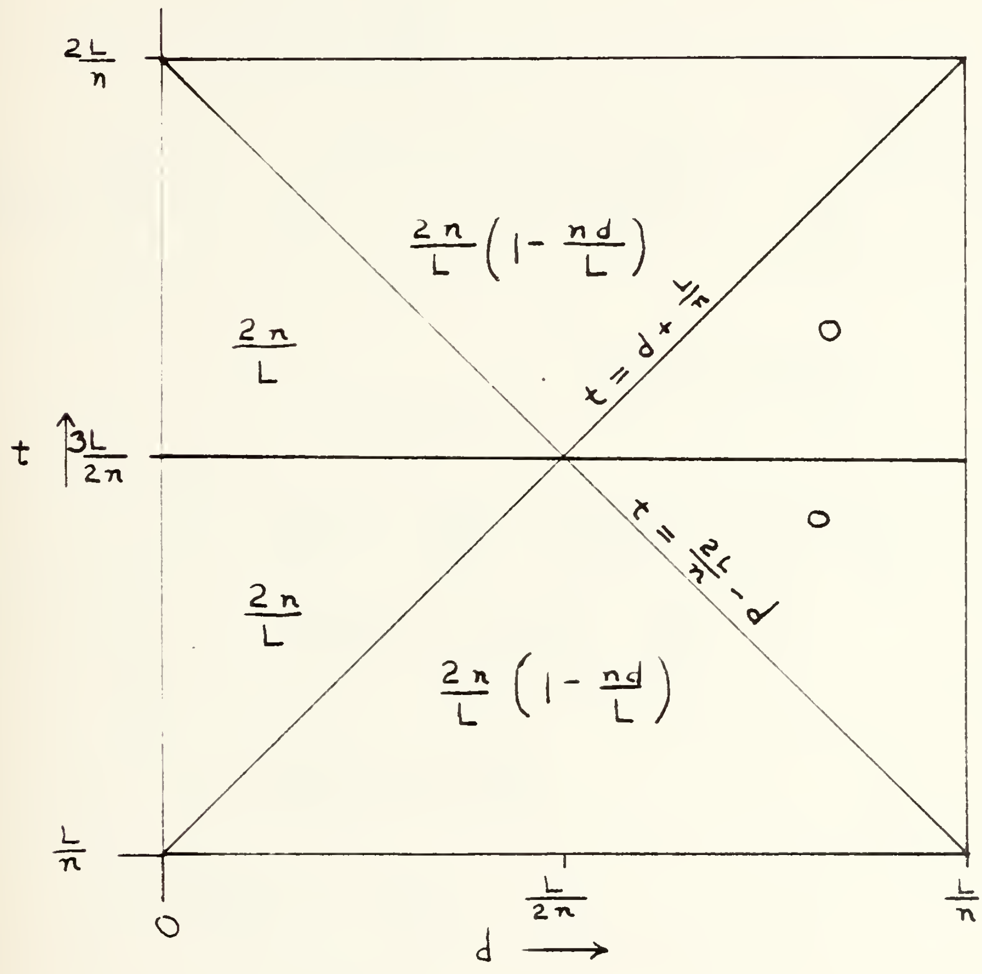


Figure 4. Unconditional probability density function

The conditional distribution of D, given S(2), is now found by taking the expectation of $f_{D|T,S(2)}(d|t)$ with

respect to the conditional distribution of T given S(2). Thus, unconditioning over T (given S(2)) has the following mathematical form:

$$f_{D|S(2)}(d) = \int_{\frac{L}{n}}^{2L/n} f_{D|T,S(2)}(d|t)\ f_{T|S(2)}(t)\ dt.$$

For $0 \leq d \leq \frac{L}{2n}$,

$$f_{D|S(2)}(d) = \int_{L/n}^{d+(L/n)} \frac{2n}{L}(1 - \frac{n}{d})(\frac{n}{L})\,dt + \int_{d+\frac{L}{n}}^{\frac{2L}{n}-d} \frac{2n}{L}(\frac{n}{L})\,dt$$

$$+ \int_{(2L/n)-d}^{2L/n} \frac{2n}{L}(1 - \frac{nd}{L})(\frac{n}{L})\,dt$$

$$= 2(\frac{n}{L})^2(1 - \frac{nd}{L})(d + \frac{L}{n} - \frac{L}{n}) + 2(\frac{n}{L})^2(\frac{2L}{n} - d - d - \frac{L}{n})$$

$$+ 2(\frac{n}{L})^2(1 - \frac{nd}{L})(\frac{2L}{n} - \frac{2L}{n} + d)$$

$$= 2(\frac{n}{L})^2(\frac{L}{n} - \frac{2nd^2}{L}).$$

For $\frac{L}{2n} \leq d \leq \frac{L}{n}$,

$$f_{D|S(2)}(d) = \int_{\frac{L}{n}}^{(\frac{2L}{n}-d)} \frac{2n}{L}(1 - \frac{nd}{L})(\frac{n}{L})\,dt + \int_{d+(\frac{L}{n})}^{\frac{2L}{n}} \frac{2L}{n}(1-nd)(\frac{n}{L})\,dt$$

$$f_{D|S(2)}(d) = 2\left(\frac{n}{L}\right)^2\left(1-\frac{nd}{L}\right)\left(\frac{2L}{n}-d-\frac{L}{n}\right)$$

$$+\ 2\left(\frac{n}{L}\right)^2\left(1-\frac{nd}{L}\right)\left(\frac{2L}{n}-d-\frac{L}{n}\right)$$

$$= 2\left(\frac{n}{L}\right)^2\left(\frac{2L}{n}-4d+\frac{2nd^2}{L}\right).$$

Therefore in any middle line segment,

$$f_{D|S(2)}(d) = \begin{cases} 2\left(\frac{n}{L}\right)^2\left(\frac{L}{n}-\frac{2nd^2}{L}\right); & 0 \le d \le \frac{L}{2n} \\ 2\left(\frac{n}{L}\right)^2\left(\frac{2L}{n}-4d-\frac{2nd^2}{L}\right); & \frac{L}{2n} \le d \le \frac{L}{n}. \end{cases} \qquad (7)$$

c. The Conditional Cumulative Distribution Function

The cdf of D conditioned by S(2) is calculated by integrating the pdf over the range (0,d) of D;

$$F_{D|S(2)}(d) = \int_0^d f_{D|S(2)}(h)\ dh\ .$$

For $0 \le d \le \frac{L}{2n}$,

$$F_{D|S(2)}(d) = \int_0^d 2\left(\frac{n}{L}\right)^2\left(\frac{L}{n} - \frac{2nh^2}{L}\right)dh$$

$$= 2\left(\frac{n}{L}\right)^2\left(\frac{Ld}{n} - \frac{2nd^3}{3L}\right)$$

and

$$F_{D|S(2)}\left(\frac{L}{2n}\right) = \frac{5}{6} .$$

For $\frac{L}{2n} \le d \le \frac{L}{n}$,

$$F_{D|S(2)}(d) = \frac{5}{6} + \int_{\frac{L}{2n}}^{d} 2\left(\frac{n}{L}\right)^2\left(\frac{2L}{n} - 4h + \frac{2nh^2}{L}\right)dh$$

$$= \frac{5}{6} + 2\left(\frac{n}{L}\right)^2\left[\frac{2Ld}{n} - 2d^2 + \frac{2nd^3}{3L} - \left(\frac{2L}{n}\right)\left(\frac{L}{2n}\right) + 2\left(\frac{L}{2n}\right)^2 - \left(\frac{2n}{3L}\right)\left(\frac{L}{2n}\right)^3\right]$$

$$= \frac{5}{6} + 2\left(\frac{n}{L}\right)^2\left[\frac{2Ld}{n} - 2d^2 + \frac{2nd^3}{3L} - \frac{7L^2}{12n^2}\right].$$

For the middle line segment,

$$F_{D|S(2)}(d) = \begin{cases} 2\left(\frac{n}{L}\right)^2\left(\frac{Ld}{n} - \frac{2nd^3}{3L}\right); & \text{for } 0 \le d \le \frac{L}{2n} \\ \frac{5}{6} + 2\left(\frac{n}{L}\right)^2\left(\frac{2Ld}{n} - 2d^2 + \frac{2nd^3}{3L} - \frac{7L^2}{12n^2}\right); & \text{for } \frac{L}{2n} \le d \le \frac{L}{n} \\ 1; & \text{for } d > \frac{L}{n} . \end{cases} \qquad (8)$$

3. <u>Right-hand Segment</u>

Development of the conditional cumulative distribution function of the minimum distance for the right-hand segment is

identical in methodology and form to the results obtained in analyzing the left-hand segment in paragraph II.B.1 above. We therefore omit the details.

C. UNCONDITIONAL CDF FOR n SHIPS ON A LINE OF LENGTH L

The cumulative distribution functions derived thus far are conditioned on S(i), i = 1,2, ..., n. Since T ~ Uniform (0,L) and each patrol segment is identically $\frac{L}{n}$ units long, the probability of T occurring in any specific segment is,

$$P[S_i] = P[T \varepsilon (\frac{iL}{n}, \frac{(i+1)L}{n})] = [(\frac{(i+1)L}{n} - \frac{iL}{n}]/L = \frac{1}{n} .$$

Therefore the unconditioned cdf is derived from (5) and (8), as follows:

$$F_D(d) = \frac{1}{n} \sum_{i=1}^{n} F_{D|S(i)}(d) .$$

For $0 \leq d \leq \frac{L}{2n}$,

$$F_D(d) = (\frac{1}{n})\{2[(\frac{n}{L})^2(\frac{2Ld}{n} - \frac{d^2}{2} - \frac{2nd^3}{3L})] + (n-2)[(\frac{n}{L})^2(2)(\frac{Ld}{n} - \frac{2nd^3}{3L})]\}$$

$$= \frac{2nd}{nL} - \frac{nd^2}{nL^2} + \frac{4n^2d^3 - 4n^3d^3}{3nL^3}$$

$$= \frac{6L^2nd - 3Lnd^2 + 4n^2d^3 - 4n^3d^3}{3L^3}$$

and

$$F_D\left(\frac{L}{2n}\right) = \frac{5}{6} - \frac{1}{12}n = \frac{10n - 1}{12n}.$$

For $\frac{L}{2n} \leq d \leq \frac{L}{n}$,

$$F_D(d) = \frac{(10n-1)}{12n} + \left(\frac{1}{n}\right)\{2[\left(\frac{n}{L}\right)^2\left(\frac{3Ld}{n} - \frac{5d^2}{2} - \frac{2nd^3}{3L} - \frac{23L^2}{24n^2}\right]$$

$$+ (n-2)(2)\left(\frac{n}{L}\right)^2\left[\frac{2Ld}{n} - 2d^2 + \frac{2nd^3}{3L} - \frac{7L^2}{12n^2}\right]\}$$

$$= \frac{(10n-1)}{12n} + \frac{4nd-2d}{L} + \frac{3nd^2-4n^2d^2}{L^2} + \frac{4n^3d^3-4n^2d^3}{3L^3} + \frac{5}{12n} - \frac{7}{6}$$

$$= \frac{12n^2L^2d-6L^2nd+9n^2d^2L-12n^3d^2L+4n^4d^3-4n^3d^3+L^3-nL^3}{3L^3n}.$$

Therefore the cumulative distribution function of the minimum of the distances from T to n patrol units is:

$$F_D(d) = \begin{cases} \dfrac{6L^2nd-3Lnd^2+4n^2d^3-4n^3d^3}{3L^3}; & \text{for } 0 \leq d \leq \dfrac{L}{2n} \\[2ex] \dfrac{12n^2L^2d-6L^2nd+9n^2d^2L-12n^3d^2L+4n^4d^3-4n^3d^3+L^3-nL^3}{3L^3n}; & \text{for } \dfrac{L}{2n} \leq d \leq \dfrac{L}{n}. \end{cases} \tag{12}$$

III. APPROXIMATIONS TO ANALYTICAL RESULTS

A. RATIONALE FOR APPROXIMATIONS

The existance of a simple, readily applicable approximation, especially one with well understood and widely known characteristics of behavior, might be useful to the military commander and planner. This thesis investigates two such approximations and evaluates the range of values for which each may be an appropriate approximation of (12).

B. POISSON PROCESS

The Poisson process as a mechanism for positioning n units on a line of length L is appealing because of the well-developed and tractable body of theory regarding the exponential distribution of the distances between units. The limitations of the approximation will be discussed after the model has been completely developed.

1. General Properties

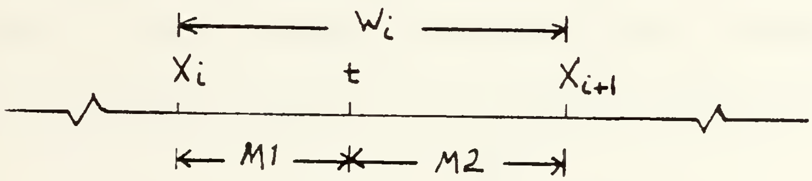
For the model let n be the average number of ships located on a line of length L, so $\lambda = \frac{n}{L}$ is the average density of ships per unit length of L. Let W_i be the distance between unit i and i+1, for all i: $i = 1,2, \ldots, n-1$. Then for a general Poisson process one has:

$$P[N = n] = \frac{(\lambda L)^n e^{-\lambda L}}{n!} .$$

The interarrival "distances" (W_i) between adjacent ships are independent exponentially distributed random variables such that

$$W_i \sim \text{Exp}(1/\lambda).$$

In what follows only those cases where T occurs between two units will be considered (the interval $(0,X_1)$ will be discussed later). Let the trouble spot T occur at random on the line of length L as illustrated below.



Let M1 represent the distance from X_i to t, M2 the distance from t to X_{i+1}, then it is easily seen that $M1 + M2 = W_i$ the distance from the i^{th} unit to the $i+1^{st}$ unit. The memoryless property of the exponential distribution implies that both M1 and M2 are distributed as exponential $(1/\lambda)$ random variables. Now with D = min(M1,M2), we have,

$$F_D(d) = P[\text{Min}(M1,M2) \le d] = P(D \le d)$$

$$= 1 - P(D \ge d) = 1 - P(M1 \ge d, M2 \ge d)$$

$$= 1 - P(M1 \ge d)P(M2 \ge d)$$

and since M1 and M2 are identically distributed,

$$F_D(d) = 1 - [1 - F_M(d)]^2 .$$

Thus one has,

$$F_D(d) = 1 - [1 - (1 - e^{-\lambda d})]^2$$

$$= 1 - e^{-2\lambda d} , \quad \lambda > 0 . \tag{13}$$

The model developed above for approximating the distribution of the minimum distance from a random trouble spot to the adjacent patrol units implies the cdf.

$$F_D(d) = 1 - e^{-\frac{2nd}{L}} ; \quad 0 \leq d \tag{14}$$

which is an exponential 2λ cdf.

2. Limitations To The Poisson Approximation

The Poisson approximation cannot guarantee that n units will be positioned within the (0,L) limit of the line to be patrolled. For example there exists a positive probability, however small, that no units will occur within (0,L).

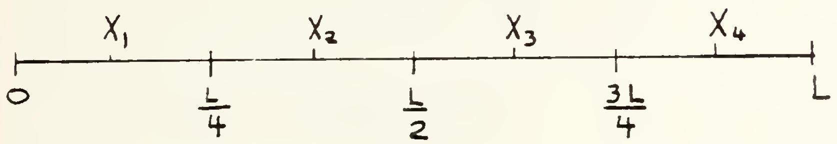
The model as developed does not consider the specific distributions involved when T occurs to the left of the location of the first unit. The effect of this simplification

is to increase the size of d for an arbitrary probability p which results in a weakening of the boundary condition developed in III.D.

The Poisson approximation does not preclude three or more units being very close to each other.

C. FIXED STATION MODEL

A second approximation, the fixed station model, is now considered. This model assumes each patrol unit maintains a fixed station located at the midpoint of the assigned patrol line segment $X_i = (2i-1)L/2n$.



1. Model Development

Since the distance between adjacent units is constant $(\frac{L}{n})$ and the distance from any unit to its patrol line segment boundaries is $L/2n$, the model developed for an arbitrary patrol segment will be representative of all segments.

Consider segment 1. Given S(1), T is distributed uniformly on $(0, \frac{L}{n})$ and the model being developed will specify the conditional cdf of $D = \min\{\frac{L}{2n} - T, T - \frac{L}{2n}\}$. That is,

$$F_{D|S(1)}(d) = P(D \le d | S(1)) = P(D \le d | T \le \frac{L}{2n}) P(T \le \frac{L}{2n})$$

$$+ P(D \le d | T > \frac{L}{2n}) P(T \ge \frac{L}{2n})$$

$$= P(\frac{L}{2n} - T \le d | T \le \frac{L}{2n}) \frac{1}{2} + P(T - \frac{L}{2n} \le d | T > \frac{L}{2n}) \frac{1}{2}$$

$$= P(T > \frac{L}{2n} - d | T < \frac{L}{2n}) \frac{1}{2} + P(T \le d + \frac{L}{2n} | T > \frac{L}{2n}) \frac{1}{2}$$

$$= \frac{1}{2}[1 - P(T \le \frac{L}{2n} - d | T < \frac{L}{2n})] + \frac{1}{2}[\frac{d + \frac{L}{2n} - \frac{L}{2n}}{L/2n}]$$

$$= \frac{1}{2}[1 - \frac{\frac{L}{2n} - d}{L/2n}] + \frac{1}{2}(\frac{2nd}{L})$$

$$= \frac{1}{2}[1 - (\frac{L-2nd}{2n})(\frac{2n}{L})] + \frac{nd}{L}$$

$$= \frac{1}{2} - \frac{L - 2nd}{2L} + \frac{nd}{L}$$

$$= \frac{2nd}{L} \; ; \quad \text{for } 0 \le d \le \frac{L}{2n} . \qquad (15)$$

Expression (15) is conditioned by the occurrence of S(1). However the conditional cdf given S(i) for each i; i = 1,2, ..., n; of $D = \min(D_i, D_{i+1})$ is identical to (15) and the summation of n conditional cdf's multiplied by the conditioning probability 1/n results in:

$$P(D \le d) = \frac{2nd}{L} \; ; \quad \text{for } 0 \le d \le \frac{L}{2n} \qquad (16)$$

which has a median value of $\frac{L}{4n}$. (17)

2. Limitations Of The Fixed Station Model

The obvious limitation of the model is the loss of realism in constraining a mobile unit to a small geographically fixed station. As a consequence of this constraint the maximum value of $D = \min(D_i, D_{i+1})$ is $\frac{L}{2n}$ for the fixed station model.

D. COMPARISON OF ANALYTIC AND APPROXIMATE MODELS

The easiest method of contrasting the two approximations with the original analytic model is through a tabulation of their individual cumulative distribution functions for specific values of n, L, and D. In Table 1 d is tabulated as a percentage of the ratio $\frac{L}{n}$. This procedure allows one to readily compare the various models at differing L and n values.

For the L = 30000, n = 6 case the distribution function for the analytic model is bounded below by that of the Poisson process and above by that of the fixed station model.

Figure (5) is a graphic presentation of the "fit" of the 2 bounding models when compared to the analytic model with parameters L = 30000 and n = 6. The graph depicts the residuals obtained from differencing the analytic and approximation models at various values of d. Data points are from Table 1.

For the case under consideration the Poisson model median value of D (0.3465) does not approximate the analytic model median (0.266) as closely as does that of the fixed station

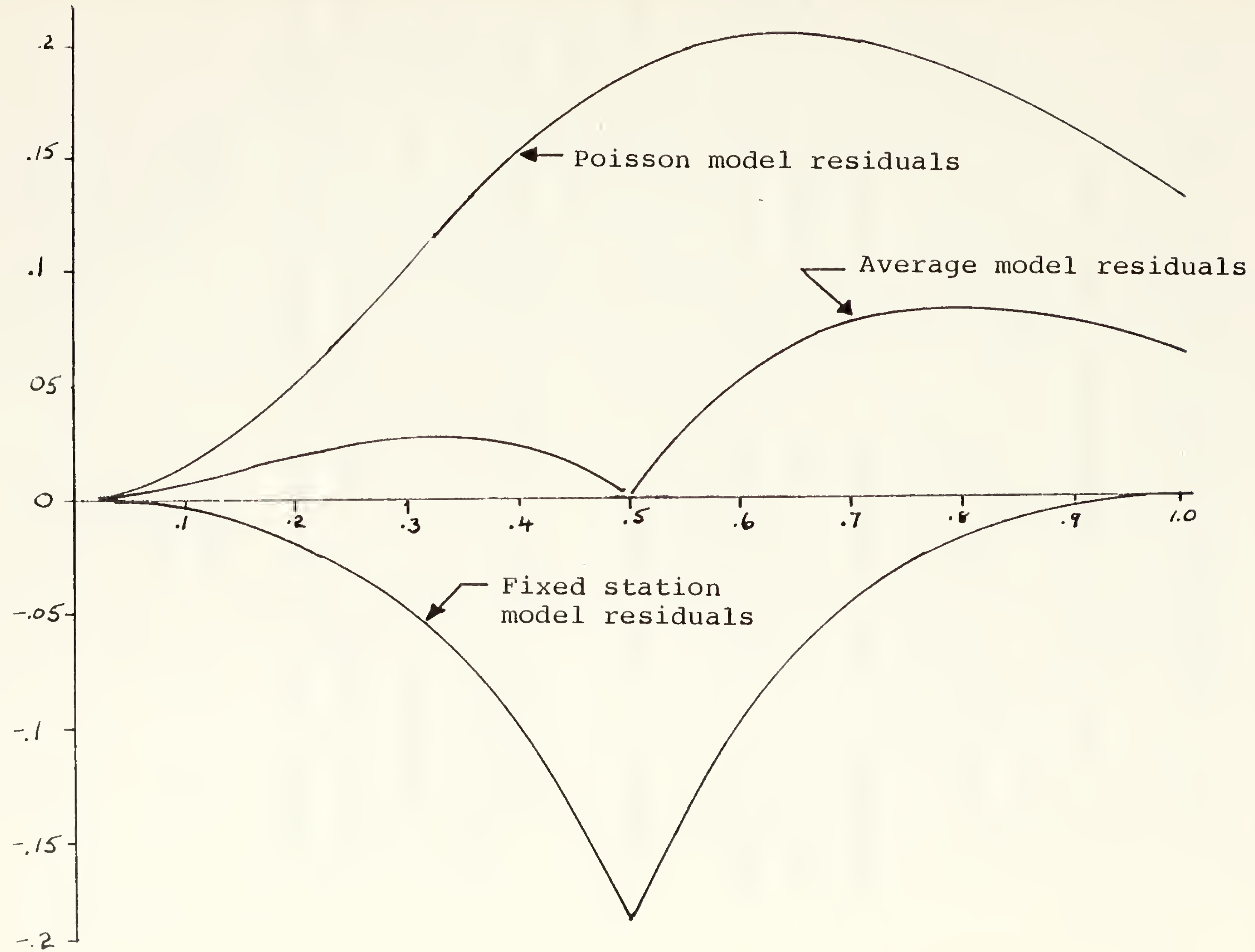
EVALUATION OF ONE DIMENSIONAL ANALYTIC AND APPROXIMATING MODELS

Model	n	L	d values*									
			.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
Poisson	-	-	.181	.33	.451	.551	.632	.699	.753	.798	.835	.865
Fixed station	-	-	.2	.4	.6	.8	1.0	1.0	1.0	1.0	1.0	1.0
Analytic	3	3	.196	.38	.546	.69	.806	.89	.946	.98	.996	1.0
Analytic	10	10	.198	.386	.559	.707	.825	.907	.958	.986	.998	1.0
Analytic	6	30000	.197	.384	.555	.702	.819	.902	.955	.984	.997	1.0
Average	-	-	.191	.365	.526	.676	.816	.85	.88	.9	.918	.932

* d is represented as a fraction of the ratio L/n, i.e. d(.2) = .2L/n

Table 1

FIGURE 5. Plot of Residuals of Approximating Models Versus Analytic (n = 6, L = 30000) model



model (0.25). Except for values in the neighborhood of $d = .5$ the Poisson process model residuals are of greater magnitude than are the fixed station residuals.

The Poisson model gives reasonable (if somewhat conservative) estimates of the analytic model results within the range $0.0 \leq d \leq .3$.

For this case, the fixed station approximation model not only bounds the analytic model, from above, it also appears to be a reasonable approximation of the analytic model over most of the range of $d \varepsilon (0.0,1.0)$. The fixed station median value of 0.25 is very close to the analytic model median value of 0.266. Using a criterion that a residual value greater than 0.1 indicates that the approximation model is inappropriate for that value of d, it is seen that the fixed station model is appropriate for all d except $0.4 < d < 0.6$. This result, while developed for the case $n = 6$ and $L = 30,000$ will hold in general except in those unusual cases where $n > L$.

The results tabulated in Table 1 suggest that there exists a natural hierarchy or ordering of the three models. If one fixes the $P(D \leq d) = .5$ and then ranks the models according to the relative magnitudes of d required by each model to obtain this probability the result is as follows:

	Model	Median
1.	Poisson	0.3465
2.	Analytic	0.266
3.	Fixed station	0.25

Arranged in this manner the models reflect the following "costs":

1. Increasing level of control by higher operational authority
2. Stochastically decreasing distances to random targets
3. Decreasing randomness of own units motion
4. In a hostile environment an increasing risk of an adversary accurately determining the positions of the patrol units
5. Decreasing probability of adjacent units being in close proximity (overlapping of coverage) which is paralleled by decreasing problems with mutual interference and contact identification.

The geometry of the residuals plotted in Figure 5 indicates that the average of the two boundary cdf's might form a more reasonable estimate of the analytic model than either of the boundary models does when considered individually.

The boundary model data in Table 1 was averaged, subtracted from the analytic (L = 30000, n = 6) model and the residuals plotted on Figure 5 as the average model residuals. The resulting residuals are obviously a very reasonable conservative approximation and lower bound for the analytic model, particularly for those $d \varepsilon (0,.65)$.

IV. APPROXIMATIONS FOR TWO-DIMENSIONAL PATROL AREAS

The previous chapters have developed a one-dimensional patrol model and two approximations which form upper and lower bounds for the analytic model. It seems a reasonable supposition that when the problem scenario is expanded to include units patrolling a two-dimensional area that a two-dimensional Poisson field would form a lower bound of the two-dimensional analytic model and that a two-dimensional lattice of fixed stations could reasonably be postulated as forming an upper bound on the two-dimensional analytic model.

The algebra involved in developing the two-dimensional analytic model for n randomly patrolling units is quite involved and time constraints precluded its development in this thesis. However the two-dimensional approximations are tractable and are presented here.

A. TWO-DIMENSIONAL POISSON FIELD

1. General Theory For A Poisson Field

Consider an array of points distributed over a plane P. For each subset B of P let N(B) be the number of points within B. The array is said to be distributed in accordance with a Poisson process with density ζ if the following assumptions are met:

(a) the number of points in non-overlapping areas are independent random variables

(b) for any subset B of finite area, N(B) is Poisson distributed with mean ζb, where b is the area of B. [Ref. 1]

The dimensional analysis of ζ is $\frac{\text{points}}{\text{length squared}}$. Thus if N(b) represents the number of points in a region with area b:

$$P(N(b) = n) = \frac{(\zeta b)^n e^{-\zeta b}}{n!}, \tag{20}$$

The probability that there are no points within a radius r of a given location is determined by

$$P(R > r) = P(\text{no points within a region of area } \pi r^2)$$

$$= e^{-\pi r^2 \zeta} . \tag{21}$$

2. Poisson Field Model

a. Assumptions and Symbology

The following assumptions are made:

(1) The n units are distributed stochastically by a Poisson process with rate ζ.

(2) The trouble spot T is equally likely to occur anywhere within the patrolled area.

(3) The area to be patrolled is L units in length and W units in width (L > W).

b. Development

In accordance with the theory for a Poisson field, $\zeta = \frac{n}{LW}$ and following expression (21), the Poisson field model is,

$$P(D \le d) = 1 - e^{-\pi d^2 \zeta}$$

$$= 1 - e^{-\pi d^2 n/LW} . \qquad (22)$$

c. Limitations to the Poisson Field Model

The Poisson field model cannot guarantee that n units will be positioned within the patrol area. Nor can the model preclude five or more units from being assigned positions which are in very close proximity to each other.

B. FIXED LATTICE OF POINTS

The cdf of the model which formed the upper bound of the one-dimensional analytic model was a series of patrol stations fixed in the middle of each patrol line segment. A reasonable two-dimensional approximation would seem to be an array of patrol stations fixed in the centers of their respective patrol areas. It is hypothesized that this two-dimensional array is an upper bound for the two-dimensional analytic model.

1. Assumptions

Assume n points are located within an L by W $(L > W)$ area as shown in Figure 6 below.

Assume that the location of T is uniformly distributed over the L by W rectangle. The n ships are to maintain station at the center of their respective $L/\sqrt{n}$ by $W/\sqrt{n}$ areas.

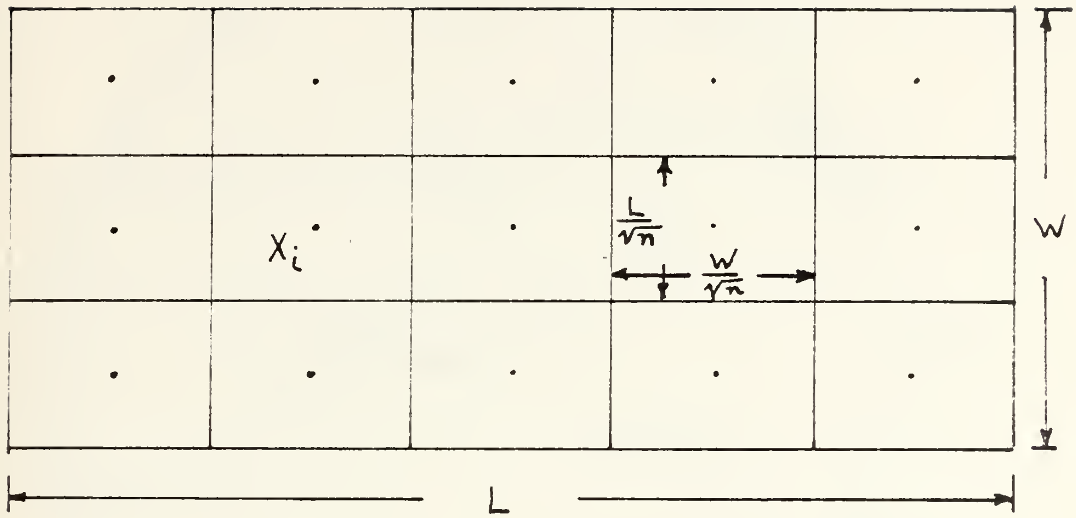


Figure 6. Fixed lattice of points

2. Development

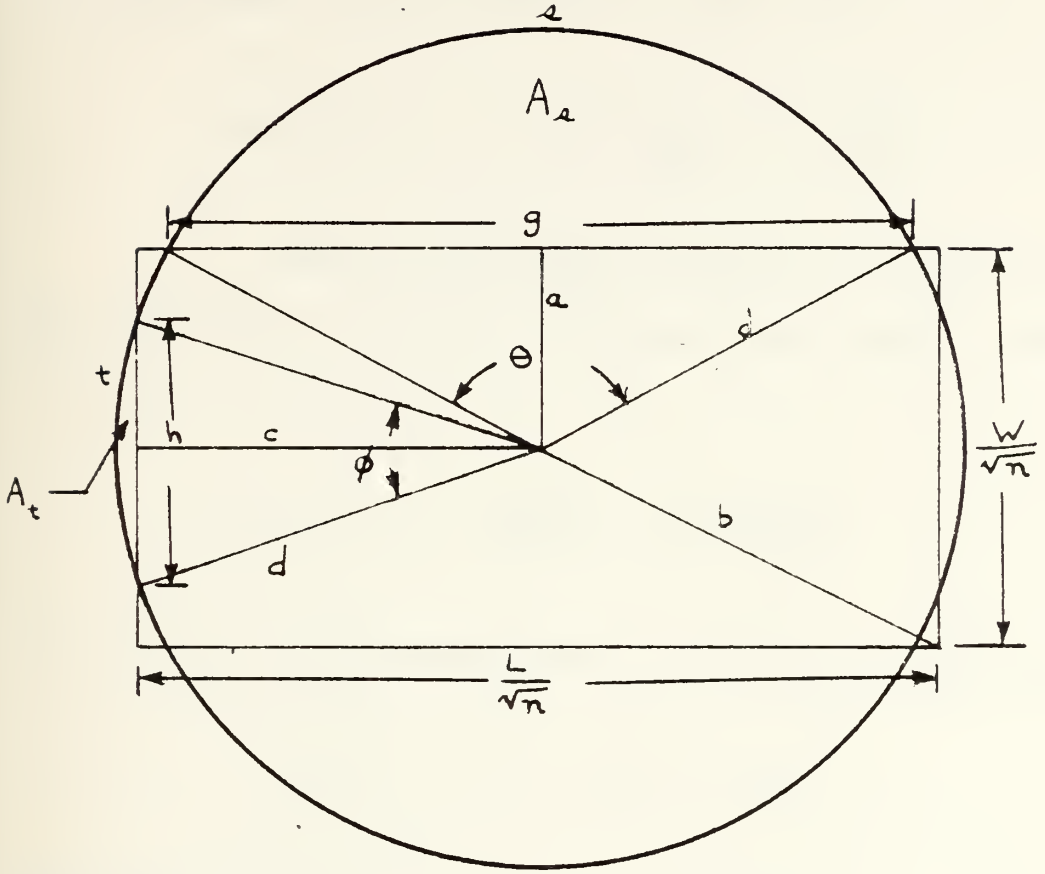
Since each of the n ships is centered in an identically shaped area the model developed for any one unit will hold true for all units. The maximum value of D is the distance from the center to the corner of a unit's patrol area as shown in Figure 7.

a. Conditional Cumulative Distribution Function

Define event S(i) as the event T ε (region i) with P[S(i)] = 1/n, and

$$F_{D|S(i)}(d) = P(D \leq d | S(i)).$$

Patrol Area X_i of Fixed Lattice Model



length g $= 2(d^2 - a^2)^{1/2} = 2(d^2 - \frac{W^2}{4n})^{1/2}$

length h $= 2(d^2 - c^2)^{1/2} = 2(d^2 - \frac{L^2}{4n})^{1/2}$

length a $= \frac{W}{2\sqrt{n}}$, length b $= (\frac{W^2}{4n} + \frac{L^2}{4n})^{1/2}$, length c $= \frac{L}{2\sqrt{n}}$

length d = d, the radius of the expanding circle,

$\theta = 2 \cos^{-1} \frac{a}{d}$, $\phi = 2 \cos^{-1} \frac{c}{d}$,

arc length s $= d\theta$, arc length t $= d\phi$

A_s = area of the segment under s but outside the rectangle

$= \frac{1}{2}(ds - ga) = d^2 \cos^{-1}(\frac{W}{2d\sqrt{n}}) - (d^2 - \frac{W^2}{4n})^{1/2}(\frac{W}{2\sqrt{n}})$

A_t = area of segment under t but outside the rectangle

$= \frac{1}{2}(dt - hc) = d^2 \cos^{-1}(\frac{L}{2d\sqrt{n}}) - \frac{L}{2\sqrt{n}}(d^2 - \frac{L^2}{4n})^{1/2}$

Figure 7

Refer to Figure (7) and consider the case of the inscribed circle of radius d such that, $0 \le d \le W/2\sqrt{n}$,

$$P(D \le d | S(i)) = (\pi d^2)\left(\frac{1}{\frac{LW}{n}}\right) = \frac{n\pi d^2}{LW} .$$

Consider the case of $W/2\sqrt{n} \le d \le L/2\sqrt{n}$ where the expanding circle has intersected the longer sides of the patrol rectangle,

$$F_{D|S(i)}(d) = \frac{n\pi d^2}{LW} - (2A_s)\left(\frac{1}{LW/n}\right).$$

As the circle radius continues to increase, consider

$$L/2\sqrt{n} \le d \le \left(\frac{W^2}{4n} + \frac{L^2}{4n}\right)^{1/2} ,$$

for which

$$F_{D|S(i)}(d) = \frac{n\pi d^2}{LW} - (2A_s + 2A_t)\left(\frac{1}{LW/n}\right).$$

Thus the conditional cdf is,

$$F_{D|S(i)}(d) = \begin{cases} \dfrac{n\pi d^2}{LW} & \text{for } 0 \le d \le \dfrac{W}{2\sqrt{n}} \\ \dfrac{n\pi d^2}{LW} - \dfrac{2nA_s}{LW} & \text{for } \dfrac{W}{2\sqrt{n}} \le d \le \dfrac{L}{2\sqrt{n}} \\ \dfrac{n\pi d^2}{LW} - \dfrac{2nA_s}{LW} - \dfrac{2nA_t}{LW}, & \text{for } \dfrac{L}{2\sqrt{n}} \le d \le \left(\dfrac{W^2}{4n} + \dfrac{L^2}{4n}\right)^{1/2} . \end{cases}$$

b. Cumulative Distribution Function

To uncondition over the event S(i) one needs to take the following summation,

$$F_D(d) = \frac{1}{n} \sum_{i=1}^{n} F_{D|S(i)}(d)$$

$$= F_{D|S(i)}(d),$$

since the n conditional cdf's are identical. Therefore it is seen that:

$$F_D(d) = \begin{cases} \dfrac{n\pi d^2}{LW}, & \text{for } 0 \le d \le \dfrac{W}{2\sqrt{n}} \\ \dfrac{n\pi d^2 - 2nA_s}{LW}, & \text{for } \dfrac{W}{2\sqrt{n}} \le d \le \dfrac{L}{2\sqrt{n}} \\ \dfrac{n\pi d^2 - 2nA_s - 2nA_t}{LW}, & \text{for } \dfrac{L}{2\sqrt{n}} \le d \le \left(\dfrac{W^2}{4n} + \dfrac{L^2}{4n}\right)^{1/2}. \end{cases} \tag{23}$$

C. EVALUATION OF THE APPROPRIATENESS OF TWO-DIMENSIONAL APPROXIMATING MODELS

Expressions (22) and (23) form hypothetical bounds to an undeveloped area patrol model. If the hypothesis is correct it would seem reasonable that the two-dimensional bounding functions should behave in a manner similar to that of the one-dimensional bounding functions. That is one would not only expect the fixed array model to yield stochastically larger results than the Poisson field model but

one would also expect to see very close agreement between the two models for small values of d. Table 2 which tabulates the evaluation of the two models for parameters n = 25, L = 30000 and W = 20000, indicates that for this case the two-dimensional models are indeed well behaved.

In the one-dimensional case it was found that the average of the boundary models was itself a very reasonably estimator of the analytic model. If the hypothesis regarding the two-dimensional boundary models is correct then a similar averaging process could be considered a better approximation of the analytic area patrol model than either boundary model is when considered separately. The average of the two-dimensional boundary models is also tabulated in Table 2.

Model	d values* (in units of length)								
	0	500	1000	1500	2000	2500	3000	3500	3605
Poisson field	1	.032	.123	.255	.408	.559	.692	.798	.818
fixed point array	1	.033	.131	.295	.524	.733	.920	.998	1.0
average	1	.032	.127	.275	.466	.646	.806	.898	.909

* for parameters n = 25, L = 30,000, W = 20,000.

Table 2. Evaluation of Two-dimensional approximating models

V. ANALYTICAL MODEL WITH RADIUS OF INFLUENCE

The initial analytic model, equation (12), was developed under the assumption that an aircraft carrier be considered a point on its patrol line segment. This restriction effectively eliminated from the previous analysis the known ability of a carrier's airwing to exercise considerable influence over the environment adjacent to the carrier. We now modify the model to allow an examination of the contribution to $P(D \leq d)$ made by an airwing controlling a radius r on either side of the carrier's position.

An additional benefit of relaxing the zero radius of influence assumption may be a quantitative model of how a units' probability of being within a specified distance of a random target will vary as the radius of influence is changed.

A. ASSUMPTIONS

All the assumptions made for the initial analytic model remain in effect with the exception of assumption II.A.(5) which constrained a carrier to being considered a point on the patrol line segment. In this modification each carrier will be considered to have a radius of influence, r, on either side of the carrier's position. If the trouble spot T occurs within $\pm$ r of the carrier's position the distance from the carrier to the trouble spot is defined as zero. Distance to the

trouble spot, T, will be measured to the edge of a carrier's radius of influence instead of to the carrier's actual position as was done in Chapter II.

There are no restrictions regarding the movement of a carrier along its patrol line segment. This means that the second carrier would not be restricted from periodically exerting influence over a portion of the first r miles of the third carrier's line segment (see X_2' in Figure (8)).

B. MODEL DEVELOPMENT

1. Cumulative Distribution Function

In the initial analytical model,

$$D = \min(D_1, D_2, \ldots, D_n) \quad \text{where} \quad D \in (0, \tfrac{L}{n}).$$

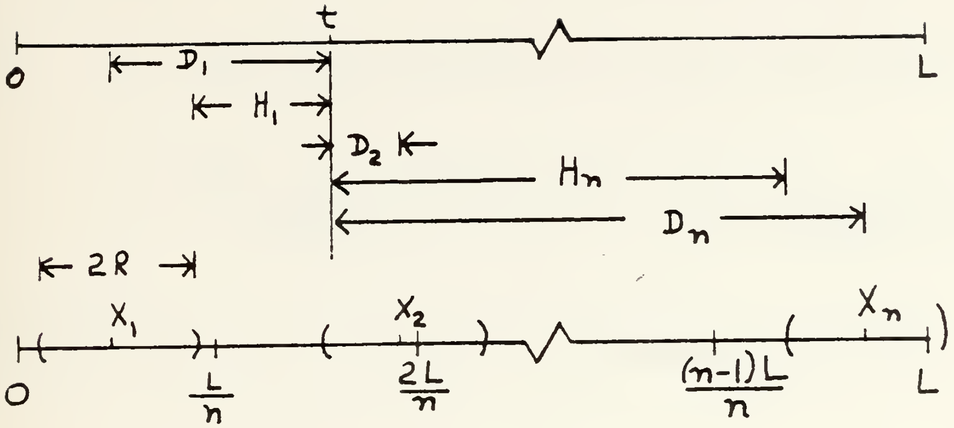
In the current modification

$$H = \min(H_1, H_2, \ldots, H_n) \quad \text{where} \quad H \in (0, \tfrac{L}{n} - r)$$

and since $D_i = H_i + r$ for $i = 1, 2, \ldots, n$ then

$$H = \begin{cases} D-r & \text{for} \quad r \leq D \leq \frac{L}{n} \\ 0 & \text{for} \quad 0 \leq D \leq r. \end{cases}$$

Now for $h > 0$,



X_i, $i=1,2,\ldots,n$	the position of the i^{th} carrier
r	the radius of influence on either side of a carrier's position
L	length of the line to be patrolled
n	the number of carriers patrolling the line
t	location of a randomly occurring trouble spot
H_i, $i=1,2,\ldots,n$	the distance measured from the random trouble spot to the nearest extremity of the i^{th} carrier's radius of influence

NOTE: $D_i = H_i + r$

FIGURE 8. Schematic and Symbology for the Radius of Influence Model

$$F_H(h) = P[H \le h] = P[H \le h | D \ge r]P(D \ge r)$$

$$+ P[H \le h | D \le r]P(D \le r)$$

$$= P[D-r \le h | D \ge r]P[D \ge r] + P[0 \le h | D \le r]P[D \le r]$$

$$= \frac{P[D-r \le h, D \ge r]P[D \ge r]}{P[D \ge r]} + \frac{P[0 \le h, D \le r]P[D \le r]}{P[D \le r]}$$

and since $[0 \le h]$ is certain, the above reduces to

$$P[r \le D \le h+r] + P[D \le r] = F_D(h+r) - F_D(r) + F_D(r)$$

$$= F_D(h+r); \quad 0 \le h \le \frac{L}{n} - r. \tag{24}$$

The radius of influence model is

$$F_H(h) = F_D(h+r) \quad \text{for} \quad 0 \le h \le \frac{L}{n} - r \tag{25}$$

which is the initial analytic model [expression (12)] evaluated at $D = h + r$.

2. Marginal Rate of Return With Respect to Radius of Influence Model

When the radius of influence model is differentiated with respect to r the result is the equivalent to evaluating the pdf of equation (12) at h+r:

$$f_H(h) = f_D(h + r) \tag{26}$$

At $r = 0$ this equation is the true pdf associated with expression (12); at values of $r > 0$ this function gives the expected marginal return in terms of $P(H \leq h)$ for a unit change in the radius of influence.

C. MODEL EVALUATION

The following plots were developed to assist in visualizing two specific realizations of the cumulative distribution function and marginal rate function for the radius of influence model.

L	n	Type of plot	Figure
10	10	cdf	9
10	10	marginal rate	10
30000	6	cdf	11
30000	6	marginal rate	12

The parameters $L = 30{,}000$ miles and $n = 6$ ships were chosen as being representative of the imaginary but roughly realistic situation described in the Introduction.

The plot in Figure 11 appears to be quite linear for all values of r up to and in some cases beyond the $r = 0.0$ median value of 1330 miles. It is also of interest to note that for large ratios of $\frac{L}{n}$, that for a given h a constant increase in r results in a near constant increase in $P(H \leq h)$. This characteristic is not observed in the $L/n = 1$ case.

By the time the distance from the target has reached 3500 miles the $n = 6$, $L = 30000$ cdf has exceeded 0.95. In

other words, for all values of $r \geq 0$ $P(H \leq 3500) > .95$. While this may be a counter-intuitive result it is not unreasonable if one remembers that a value of $h = 3500$ miles encompasses a $7000 + 2r$ mile portion of the total line.

D. EMPLOYING THE RADIUS OF INFLUENCE MODEL

We consider some examples which illustrate use of the model. The operational commander may desire to know the minimum number of units necessary to patrol a 30000 nautical mile coast such that the nearest unit has at least a 60% chance of being within a $2\frac{1}{2}$ day transit of a random spot along the coast. Further he may desire to know if there is any substantial advantage to loading the carriers with Type A aircraft (180 nautical mile radius) as opposed to loading with Type B (300 nautical mile radius). The units steam at 25 kts.

Solve equation (25) for n at $h = 1500$, $r = 180$, $L = 30000$ and $P(H \leq h) \geq .6$. The result is $n \approx 6$. Equation (25), evaluated at $n = 6$, $L = 30000$, $h = 1500$, $r = 180$ results in

$$F_H(h) = 0.611 \geq 0.6,$$

as required.

The marginal rate of return function (26), with the above conditions gives

$$f_H(h) = 0.0003,$$

which indicates one could expect a 0.0003 increase in $F_H(h)$ for each unit increase in r. The Type B aircraft's 300 nautical mile radius will increase $F_H(h)$ by approximately $(300 - 180)(0.0003) = 0.036$ over the value of 0.611. (The actual increase by solving expression (25) for both R = 180 and R = 300 is 0.03959.)

As a result an operational planner would inform the commander that it would require a minimum of six carriers to patrol the line. Additionally, based on an MOE of $P(D \leq d)$, there is no substantial advantage to be gained by loading a particular type aircraft on the six carriers.

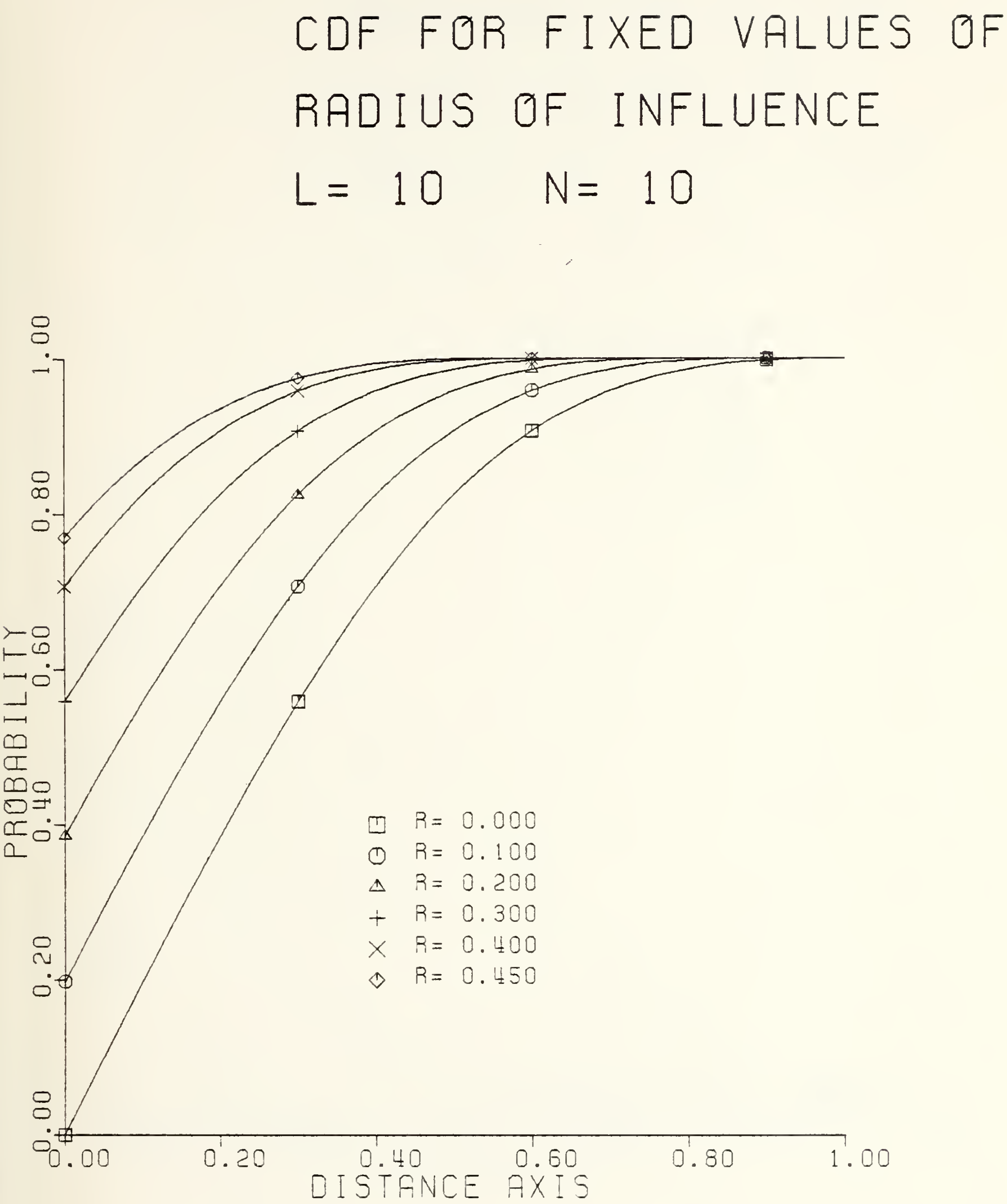

CDF FØR FIXED VALUES ØF
RADIUS ØF INFLUENCE
L= 10   N= 10
PRØBABILITY
0.00
0.20
0.40
0.60
0.80
1.00
DISTANCE AXIS
0.00
0.20
0.40
0.60
0.80
1.00
R= 0.000
R= 0.100
R= 0.200
R= 0.300
R= 0.400
R= 0.450

FIGURE 9

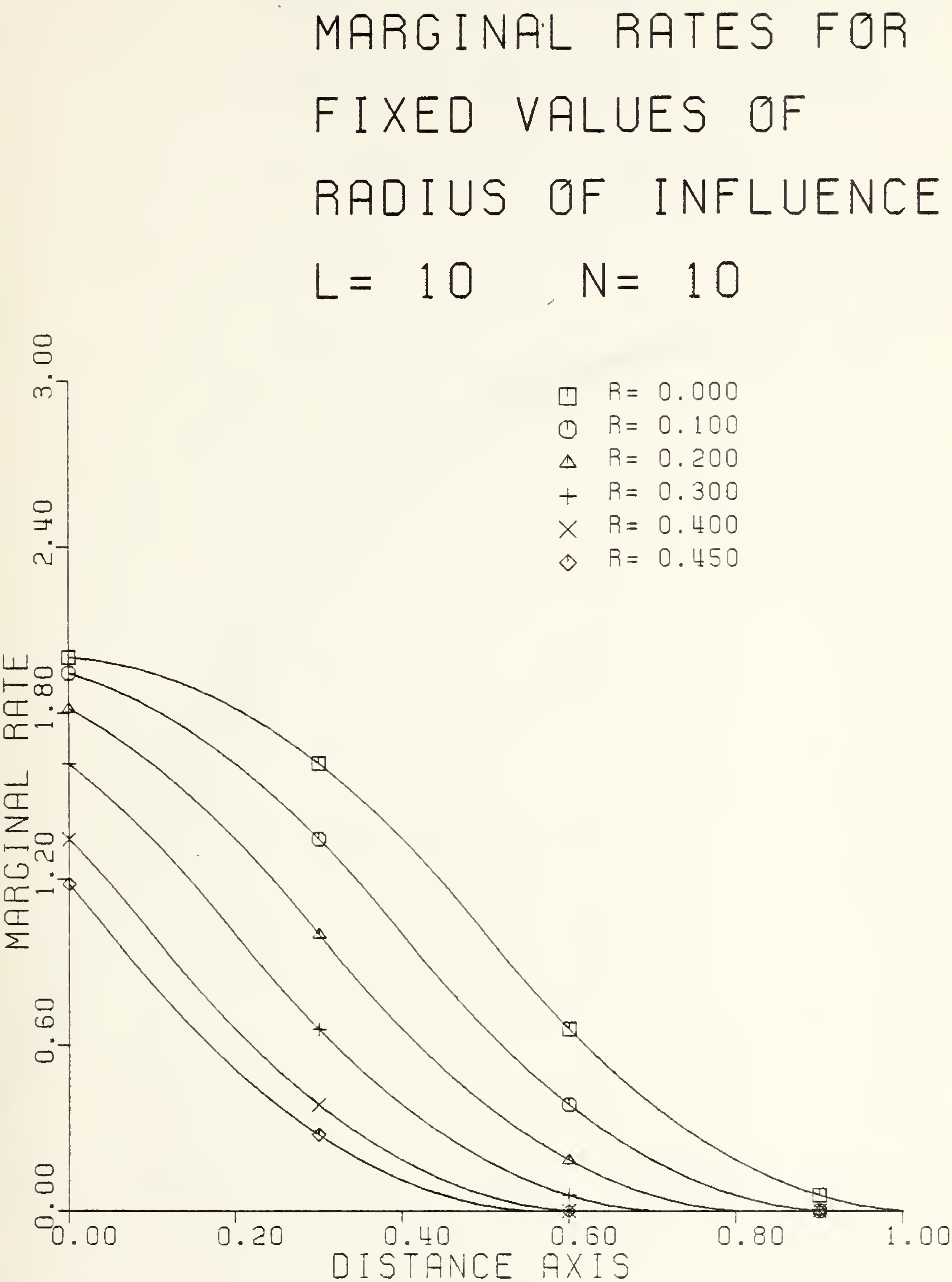

MARGINAL RATES FOR
FIXED VALUES OF
RADIUS OF INFLUENCE
L= 10 N= 10
R= 0.000
R= 0.100
R= 0.200
R= 0.300
R= 0.400
R= 0.450
MARGINAL RATE
0.00
0.60
1.20
1.80
2.40
3.00
0.00
0.20
0.40
0.60
0.80
1.00
DISTANCE AXIS

FIGURE 10

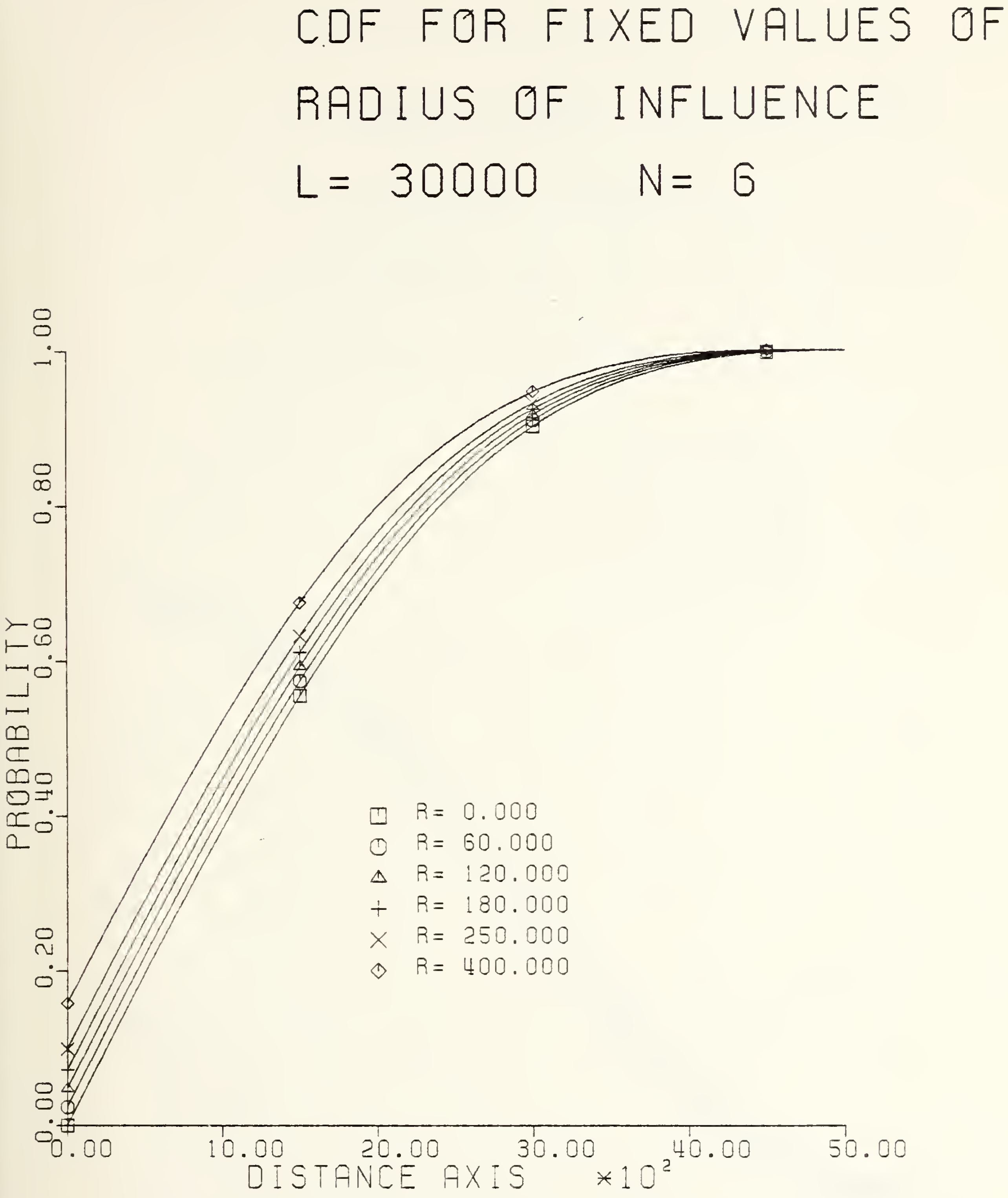

CDF FØR FIXED VALUES ØF
RADIUS ØF INFLUENCE
L= 30000 N= 6
PRØBABILITY
0.00
0.20
0.40
0.60
0.80
1.00
R= 0.000
R= 60.000
R= 120.000
R= 180.000
R= 250.000
R= 400.000
0.00
10.00
20.00
30.00
40.00
50.00
DISTANCE AXIS $\times 10^2$

FIGURE 11

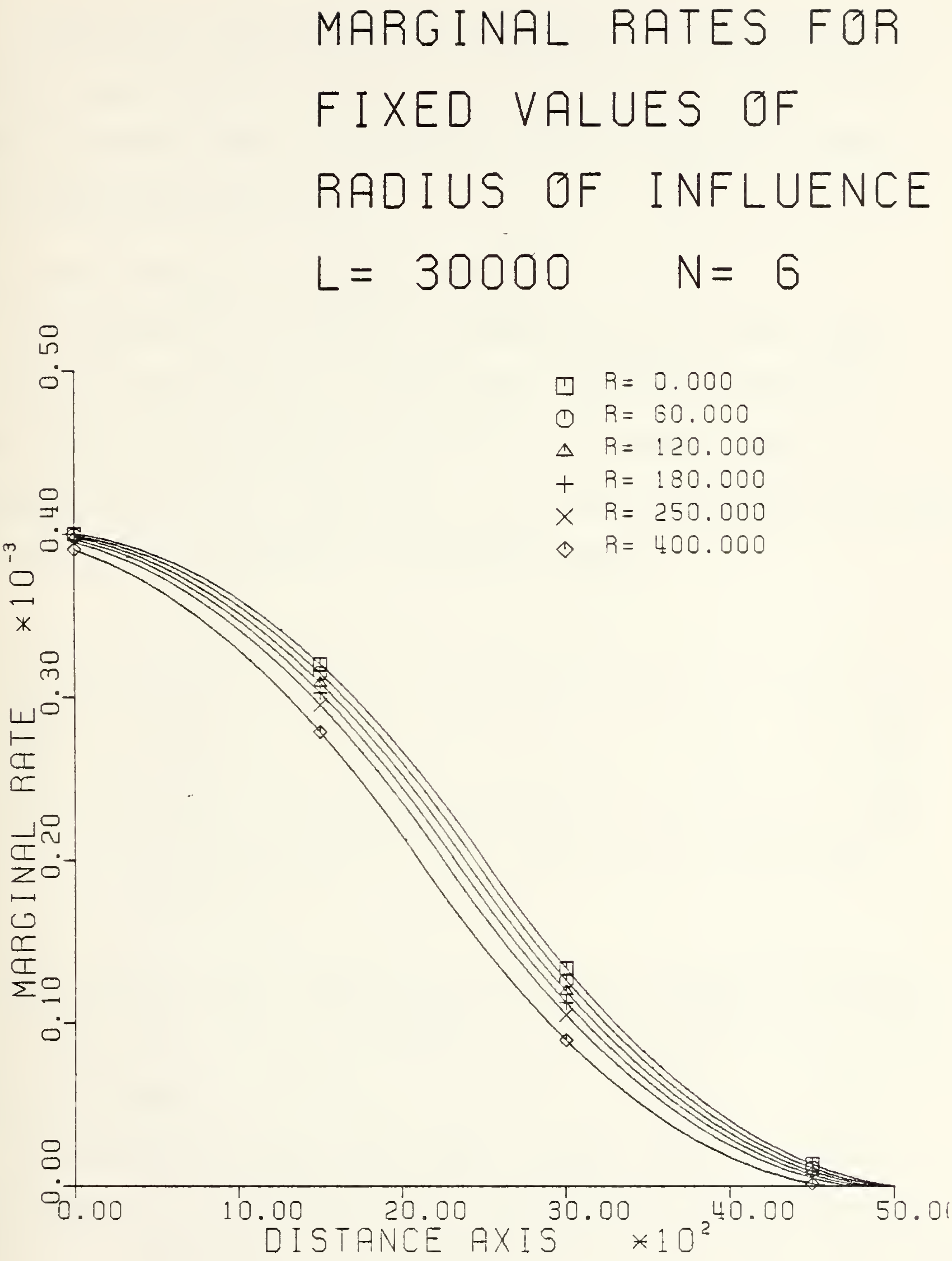

MARGINAL RATES FOR
FIXED VALUES OF
RADIUS OF INFLUENCE
L= 30000 N= 6
R= 0.000
R= 60.000
R= 120.000
R= 180.000
R= 250.000
R= 400.000
MARGINAL RATE ×10^-3
0.00
0.10
0.20
0.30
0.40
0.50
0.00
10.00
20.00
30.00
40.00
50.00
DISTANCE AXIS ×10^2

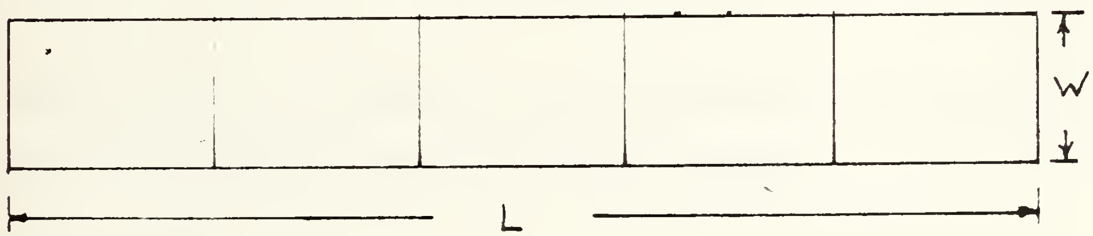
FIGURE 12

VI. EXTENSIONS OF THE ANALYTIC MODELS

During the course of the development of this thesis several potentially interesting extensions were discussed. Some of these ideas are presented below as possible areas of future investigation.

A. MULTI-DIMENSIONAL PATROL AREAS

The assumption limiting a carrier to patrolling along a line is unnecessarily restrictive. A natural extension would be to expand the patrol line to a patrol area of length L and width W such that each carrier would be assigned a L/n x W patrol area as illustrated below:



T, the trouble spot, would be restricted to either occurring along the (0,L) line, or perhaps to occurring anywhere within the (L,W) area.

An extension which moves even farther from the patrol line model, but which naturally follows the previous extension is to consider a grid imposed on an open ocean area with

one carrier assigned each rectangular patrol area. T would occur randomly anywhere within the patrol area boundaries.

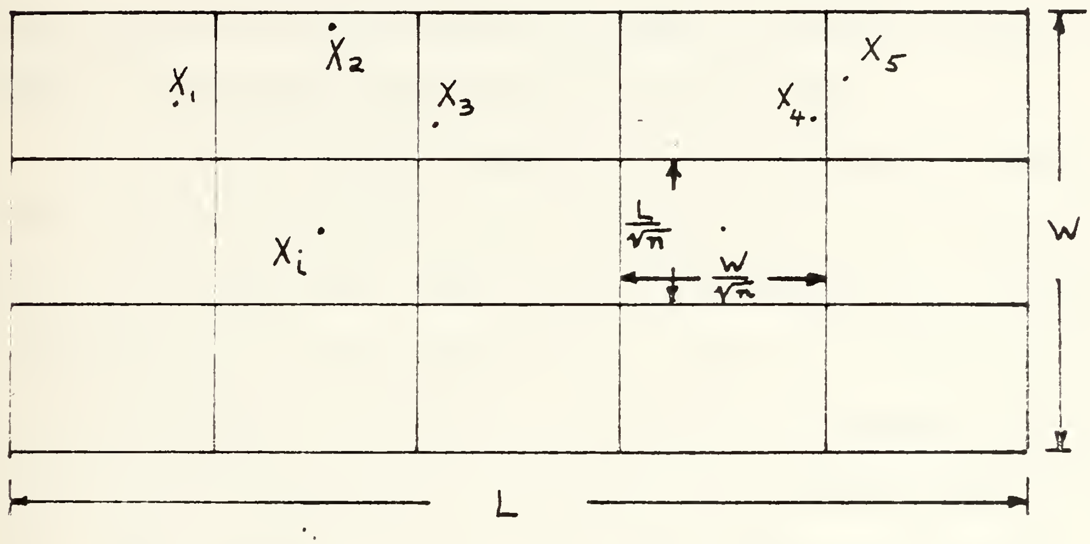


Figure 13. n carriers randomly patrolling a 2 dimensional area.

The radius of influence of each carrier could also be incorporated into the two-dimensional models.

B. GAME THEORY

The stipulation that the location of the trouble spot is a random variable with a particular probability distribution may be realistic in most situations involving a naval presence mission. Where an adversary relationship can be postulated this stochastic assumption as well as the assumption of independence between the location of T and the positions of the carrier units (X_i's) may be untenable. If

the adversary is capable of some degree of knowledge of the carrier's positions, through radio direction finding, shadowing or satellite reconnaissance, he is capable of choosing T so as to maximize his return (perhaps by maximizing the expected distance to the patrolling force). In this case the problem takes on a game theoretic aspect, and results from game theory might be applicable.

If, through careful analysis of an adversary's intelligence gathering capabilities, force levels and historic political intentions, one is able to determine an a priori distribution of the location of T, then it is possible to use the probability integral transform to transform the a priori distribution into a Uniform (0,1) distribution. This might enable one to utilize the models developed herein and then transform the results back to the original a priori space.

C. ECONOMIC COST OF VARYING THE RADIUS OF INFLUENCE

There are economic costs involved in varying the radius of influence of a carrier. Through the use of the radius of influence model developed herein and basic cost considerations, an analysis of the cost tradeoffs involved in achieving a particular improvement in $P(D \leq d)$ could be developed. This development could include economic tradeoffs in terms of various combinations of:

1. Increasing a carrier's radius of influence
2. Increasing the transit speed of the carrier

3. Increasing the number of carriers

4. Increasing intelligence concerning the location of T.

LIST OF REFERENCES

1. Parzen, E., Stochastic Processes, p. 31-33, Holden-Day, Inc., 1962.

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The distribution of the minimum distance between a random target and units patrolling along a line.